

- Complete Integrals: If from the p.d.e.

$$f(x, y, z, p, q) = 0$$

We can find a relation

$$F(x, y, z, a, b) = 0$$

which contains as many arbitrary constants as there are independent variables, the relation $F(x, y, z, p, q) = 0$ is known as Complete integrals of the given equation.

- Particular Integral: Giving the particular values to the constants a and b occurring in the complete integral, we obtain a solution (of the given p.d.e.) which we shall call as Particular Integral.

- Singular Integral: If

$$F(x, y, z, a, b) = 0$$

is the complete integral. Then the singular integral is obtained by eliminating a and b from $f = 0$, $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0$.

- General Integral: If two functions u and v of x, y, z are connected by an arbitrary function $f(u, v) = 0$ then the eliminating f we get a p.d.e. of the form $p\phi + q\psi = R$.

The solution of the equation is $f(u, v) = 0$ which is called the General Integral of the equation.

• Linear P.D.E. of order one:—

A differential eqn involving partial derivatives p and q only and no higher is called of order one. If the degree of p and q are unity then it is called linear p.d.e. of order one.

• Lagrange's Linear Equation:—

The p.d.e. of the form $Pp + Qq = R$

where P, Q, R are fns of x, y, z is the standard form of the linear p.d.e. of order one and is called Lagrange's Linear Equation.

Lagrange's Solution of linear p.d.e. :-

Solution:-

The general solution of the linear p.d.e. $Pp + Qq = R \dots \dots \dots \dots \dots \dots \quad (1)$

where P, Q, R are fns. of x, y, z .

is $\phi(u, v) = 0 \dots \dots \dots \dots \quad (2)$

where ϕ is an arbitrary function and $u(x, y, z) = C_1$ and $v(x, y, z) = C_2 \dots \dots \dots \quad (3)$

form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \dots \dots \dots \quad (4)$$

We have, by eliminating ϕ from (2) we arrive at (1). Hence (2) is general solution of (1) and we now proceed to obtain u and v for substitution in (2).

Differentiating (3), we obtain total diff. eqn. i.e.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \text{and}$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

Solving above equations by using cross-multiplication for dx, dy, dz , we have

$$\begin{aligned} \frac{dx}{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}} &= \frac{dy}{\frac{\partial v}{\partial z} \cdot \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial z}} \\ &= \frac{dz}{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}} \quad \dots \dots \dots (5) \end{aligned}$$

$$\text{i.e. } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Equations (4) are called Lagrange's auxiliary (or subsidiary) equations of (1).

Therefore, if $u=a$ and $v=b$ are two independent solutions of the system of differential eqns

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{then } \phi(u, v) = 0 \quad \text{is a solution of } Pp + Qq + Rr = R, \quad \phi \text{ being arbitrary fn.}$$

① First convert the given linear p.d.e. of first order in standard form (Lagrange's eqn.)

$$Pp + Qq = R \dots \dots \dots (1)$$

② Find the Lagrange's auxiliary eqns as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \dots \dots \dots (2)$$

③ Find the independent solution

$$u(x, y, z) = a \text{ and } v(x, y, z) = b \text{ of (2)}$$

④ Then we obtain the general solution of (1) in one of the following three equivalent forms:

$$(i) F(u, v) = 0$$

$$(ii) u = F(v)$$

$$(iii) v = F(u)$$

F being an arbitrary function.

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